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B.Sc-part-I(H). paper-I, Date-27-04-2020
Group (Algebra)

Q.1. Define a group and prove that identity element in a group is unique.

Soln Group: → A non-empty set set G, together with a binary operation 'o' is called a group (G, o) if the following laws hold for the operation 'o' -

I Closur law: when G is closed under the binary operation 'o' holds. i.e. $aob \in G \ \forall a, b \in G$.

II Associative law: The associative law for the operation 'o' holds i.e. $(aob)oc = ao(boc), \forall a, b, c \in G$

III Existence of Identity Elements: The identity element exists in G i.e. there exists an abelian $e \in G$ such that
$$aoe = eoa = a \ \forall a \in G.$$

IV Existence of inverse Elements: - The inverse elements of all the elements of G exist i.e. for every $a \in G$ there exists an element $b \in G$ such that $aob = boa = e$, e being the identity element

The identity element of o is denoted by $\bar{a}'$ and so
$$a o \bar{a}' = \bar{a}' o a = e$$

Proof: Let (G, o) be a group, with the identity element $e \in G$.
We have to prove that identity element is unique.
If possible let $e' \in G$ be another identity element of the group
As e is the identity element, by definition
$$aoe = eoa = a \ \forall a \in G$$
As $e' \in G$, putting e' for a in the above, we get
$$e'oe = eoe' = e' \quad \text{——} (1)$$
By assumption e' is also an identity element of G
∴ By definition, $aoe' = e'oa = a$
As $e \in G$, $\quad eoe' = e'oe = e$ —— II
From I and II $e = e'$. So e is the unique identity element.

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Q.2. If a and b are any two elements of a group then prove that $(ab)^{-1} = b^{-1}a^{-1}$, where a^{-1}, b^{-1} and $(ab)^{-1}$ respectively stand for inverse of a, b and ab in G.

Proof: We have to prove that the inverse element of $a \circ b$ is $b^{-1} \circ a^{-1}$, where $a, b \in G$.

By definition of inverse element, the inverse element of x is y if $x \circ y = y \circ x = e$, the identity element.

So, we have to prove that

$$(a \circ b) \circ (b^{-1} \circ a^{-1}) = (b^{-1} \circ a^{-1}) \circ (a \circ b) = e$$

Now, $(a \circ b) \circ (b^{-1} \circ a^{-1}) = a \circ (b \circ b^{-1}) \circ a^{-1}$ [$\because$ associative law holds for a group]

$= a \circ e \circ a^{-1}$ [$\because b \circ b^{-1} = e$]

$= (a \circ e) \circ a^{-1} = a \circ a^{-1} = e$

$(b^{-1} \circ a^{-1}) \circ (a \circ b) = b^{-1} \circ (a^{-1} \circ a) \circ b$

$= b^{-1} \circ e \circ b = b^{-1} \circ (e \circ b)$

$= b^{-1} \circ b = e$

Thus, we get, $(a \circ b) \circ (b^{-1} \circ a^{-1}) = (b^{-1} \circ a^{-1}) \circ (a \circ b) = e$

So, $b^{-1} \circ a^{-1}$ is the inverse element of $a \circ b$

i.e. $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ proved.

Q.3. If G is a group and $a, b \in G$ then prove that $(ab)^2 = a^2b^2$ if and only if G is abelian.

Proof:→ Since the group G is abelian.

$\therefore$ $ab = ba$ —(I) $\forall a, b \in G$

Now, $(ab)^2 = (ab)(ab)$

$= a(ba)b$ [By associative law]

$= a(ab)b$, by I

$= (aa)(bb)$ [By associative law]

$= a^2b^2$

Conversely, we have $(ab)^2 = a^2b^2$

$\Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$ [By associative law]

$\Rightarrow ba = ab$ [By cancellation law]

$\Rightarrow$ the group G is abelian.

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Q.1. Find the condition that the line $x\cos\alpha + y\sin\alpha = p$ may touch the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$

Soln: The equation of the curve is $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ ——— (I)

Differentiating with respect to x, we get

$$m\frac{x^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$$

At the point (x_1, y_1) of the curve, $\frac{dy}{dx} = -\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}$

The equation of the tangent to the curve (I) at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{x_1, y_1}(x - x_1) = -\frac{b^m}{a^m}\cdot\frac{x_1^{m-1}}{y_1^{m-1}}(x - x_1)$$

$$\Rightarrow \frac{y_1^{m-1}}{b^m}(y - y_1) + \frac{x_1^{m-1}}{a^m}(x - x_1) = 0$$

$$\Rightarrow \frac{yy_1^{m-1}}{b^m} - \frac{y_1^m}{b^m} + \frac{xx_1^{m-1}}{a^m} - \frac{x_1^m}{a^m} = 0$$

$$\Rightarrow \frac{xx_1^{m-1}}{a^m} + \frac{yy_1^{m-1}}{b^m} = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m}$$

Since the point (x_1, y_1) lies on the curve (I) and therefore

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1 \text{ ——— II}$$

$$\therefore \frac{xx_1^{m-1}}{a^m} + \frac{yy_1^{m-1}}{b^m} = 1 \text{ ——— III}$$

By question, the eqn of the tangent is

$$\frac{x}{p}\cos\alpha + \frac{y}{p}\sin\alpha = 1 \text{ ——— (IV)}$$

Comparing III and IV, we get

$$\frac{x_1^{m-1}}{a^m} = \frac{1}{p}\cos\alpha \text{ and } \frac{y_1^{m-1}}{b^m} = \frac{1}{p}\sin\alpha$$

$$\therefore x_1 = \frac{a^{\frac{m}{m-1}}(\cos\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}} \text{ and } y_1 = \frac{b^{\frac{m}{m-1}}(\sin\alpha)^{\frac{1}{m-1}}}{p^{\frac{1}{m-1}}}$$

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Rest part of Q.1

Substituting the values of x_1 and y_1 in II, we get

$$\frac{a^{\frac{m^2}{m-1}}(\cos\alpha)^{\frac{m}{m-1}}}{a^m p^{\frac{m}{m-1}}} + \frac{b^{\frac{m^2}{m-1}}(\sin\alpha)^{\frac{m}{m-1}}}{b^m p^{\frac{m}{m-1}}} = 1$$

$$\Rightarrow (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

Which is the required condition.

Q.2. Prove that the condition that $x\cos\alpha + y\sin\alpha = p$ should touch $x^m y^n = a^{m+n}$ is $p^{m+n} m^m n^n = (m+n)^{m+n} \sin^n\alpha \cdot \cos^m\alpha$.

Soln: The equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \text{ ——— I}$$

Since, $x^m y^n = a^{m+n}$

Taking logarithm on both sides, we get

$$\log(x^m y^n) = \log(a^{m+n})$$

$$\Rightarrow \log x^m + \log y^n = (m+n)\log a$$

Differentiating both sides with respect to x, We got

$$m\frac{1}{x} + n\frac{1}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{m}{n}\cdot\frac{y}{x}$$

$\therefore$ Eqn of the tangent at (x, y) is

$$Y - y = -\frac{m}{n}\frac{y}{x}(X - x)$$

$$\Rightarrow myX + nxY = mxy + nxy$$

$$\Rightarrow myX + nxY = (m+n)xy \text{ ——— (II)}$$

Now, if $X\cos\alpha + Y\sin\alpha = p$ ——— (III) be the tangent to the curve at (x, y) then (II) and III must be identical

Comparing, $\frac{my}{\cos\alpha} = \frac{nx}{\sin\alpha} = \frac{(m+n)xy}{p}$

$\therefore \frac{mp}{(m+n)\cos\alpha} = x$ and $\frac{np}{(m+n)\sin\alpha} = y$

But $x^m y^n = a^{m+n}$, $\therefore \frac{m^m p^m}{(m+n)^m \cos^m\alpha} \times \frac{n^n p^n}{(m+n)^n \sin^n\alpha} = a^{m+n}$

$$\Rightarrow p^{m+n} \cdot m^m n^n = (m+n)^{m+n} \sin^n\alpha \cos^m\alpha$$

Proved